

Kirillov–Schilling–Shimozono bijection as energy functions of crystals

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Abstract

The Kirillov–Schilling–Shimozono (KSS) bijection appearing in theory of the Fermionic formula gives an one to one correspondence between the set of elements of tensor products of the Kirillov–Reshetikhin crystals (called paths) and the set of rigged configurations. It is a generalization of Kerov–Kirillov–Reshetikhin bijection and plays inverse scattering formalism for the box-ball systems. In this paper, we give an algebraic reformulation of the KSS map from the paths to rigged configurations, using the combinatorial R and energy functions of crystals. It gives a characterization of the KSS bijection as an intrinsic property of tensor products of crystals.

1 Introduction

In proving the Fermionic formula, the combinatorial bijection called the Kirillov–Schilling–Shimozono (KSS) bijection [11, 23, 1] plays the central role. The KSS bijection is a natural generalization of the original bijection of Kerov–Kirillov–Reshetikhin [9, 10], and gives elaborated combinatorial one to one correspondences between the set of rigged configurations and elements of tensor products of the Kirillov–Reshetikhin crystals $B^{r,s}$ which we call *paths*:

$$\phi : \text{path} \longmapsto \text{rigged configuration}.$$

Here, we are focusing on the $A_n^{(1)}$ case, and the Kirillov–Reshetikhin crystals are crystals for irreducible finite dimensional modules over quantum affine algebras [7, 6] indexed by Dynkin node $r \in \{1, 2, \dots, n\}$ and $s \in \mathbb{Z}_{>0}$. Under the KSS bijection, the statistics called cocharge is preserved and hence it gives proof of the so called $X = M$ identity. Here M is called Fermionic formula, and X is the generating function of energy functions over paths. In the case we are considering, X contains the Kostka polynomial [16] as special case. Reviews by Okado [18] and Schilling [24] give excellent introduction to the subject. The reader can find an alternative approach for the problem in Section 8 of the paper [25].

Recently, a remarkable new feature of the KSS bijection was discovered. In the paper [12], the KSS bijection is identified as the inverse scattering formalism for the celebrated example of ultradiscrete integrable systems called the box-ball systems [29, 28, 3, 2], and as an outcome of the result, algebraic alternative procedure for the calculation of the map ϕ^{-1} from the rigged configurations to the paths of the form $B^{1,s_1} \otimes \dots \otimes B^{1,s_L}$ is discovered [19]. The latter formalism is used to derive explicit piecewise linear formula for ϕ^{-1} in this case as well as general solutions for the box-ball systems [15]. These formulas are written in terms of the ultradiscrete limit of the tau functions. Here, the tau functions are common apparatus in algebraic structure theory of solitons (see, e.g., [4, 17] for textbook treatments) and they satisfy the Hirota bilinear form. Interestingly, although these tau functions come from the KP hierarchy [21, 5], its expression contains the cocharge that appear in the Fermionic formula. This reveals unexpected link between the $X = M$ identities and the soliton theory. It is also interesting to note that if we restrict these tau functions to the paths with periodicity (the periodic box-ball systems [30]), we obtain ultradiscrete limit of the classical Riemann theta functions [13, 14].

So it is natural to ask what the representation theoretical origin of the KSS bijection is. As seen previously, for the case ϕ^{-1} the problem is settled in [19] if the paths are elements of $B^{1,s_1} \otimes \dots \otimes B^{1,s_L}$, and the result has already brought many insights. In this paper, we consider the case ϕ in full generality, i.e., we consider all elements of $B^{r_1,s_1} \otimes \dots \otimes B^{r_L,s_L}$. As the result, the KSS map ϕ is reformulated by algebraic language such as combinatorial R and energy function of the crystal bases. In particular, it seems that the result of [19] do not admit straightforward generalizations, therefore our result gives alternative approach to

the problem beyond the theory given in [19].

Let us explain our result in more details. Consider an element $b \in B^{r_1, s_1} \otimes \cdots \otimes B^{r_L, s_L}$. Assume that b is an element of an affine crystal. Then we consider the following isomorphism (see the end of Section 2.1 for notations)

$$u_l[0] \otimes b_1[0] \otimes \cdots \otimes b_L[0] \simeq b'_1[-e_1] \otimes \cdots \otimes b'_L[-e_L] \otimes u_l^*[E_l]$$

under the isomorphism of affine combinatorial R . Here $u_l \in B^{k, l}$ is the highest element and $u_l^* \in B^{k, l}$. We remark that this isomorphism is nothing but time evolution of the box-ball systems [3, 2]. Then our main result (Theorem 3.3) states that by taking differences of these E_l or e_i 's, we can derive alternative algorithm of the map ϕ . This is a generalization of the procedure given in [20] where the case $A_1^{(1)}$ is treated.

The key to derive such a result is formula connecting E_l in the above and shapes of Young diagrams of the rigged configuration (Proposition 4.1). Note that by virtue of the Yang–Baxter relation, we have $E_l(b) = E_l(c)$ if b and c are isomorphic under the classical part of the combinatorial R . Therefore our scheme naturally explains why Young diagrams of rigged configuration do not change for two isomorphic paths $b \simeq c$. This is a part of result [11] (see Theorem A.2), which gives foundation for the inverse scattering formalism in [12]. From more technical point of view, the original combinatorial description of the map ϕ uses combinatorial notions such as the singular strings. However, in our scheme, these combinatorial procedures are precisely rephrased in computation of the energy function Eq.(10). Therefore we can say that our scheme simplifies the rather mysterious concept of the KSS map ϕ . Since our scheme is purely algebraic, it will be a very interesting future problem to give an algebraic alternative reformulation of the theory given in [11].

The organization of this paper is as follows. In Section 2, we recall basic facts about the Kirillov–Reshetikhin crystals, especially explicit description of the combinatorial R and energy function following Shimozono [26]. In Section 3, we give our main result (Theorem 3.3). In Section 4, we give a formula which translates the KSS bijection into the energy function. In Section 5, we prove Theorem 3.3. In Appendix, we collect necessary facts from the KSS bijection including its definition.

2 Preliminaries

2.1 Kirillov–Reshetikhin crystal

Let $W_s^{(r)}$ be a $U_q(\mathfrak{g})$ Kirillov–Reshetikhin module, where we shall consider the case $\mathfrak{g} = A_n^{(1)}$. The module $W_s^{(r)}$ is indexed by a Dynkin node $r \in \{1, 2, \dots, n\}$ and $s \in \mathbb{Z}_{>0}$. As a $U_q(A_n)$ -module, $W_s^{(r)}$ is isomorphic to the irreducible module corresponding to the partition (s^r) . For arbitrary r and s , the module $W_s^{(r)}$ is known to have a crystal base [7, 6], which we denote by $B^{r, s}$. As a set, $B^{r, s}$ contains all the column strict semi-standard Young tableaux of depth r and width s over the

alphabet $\{1, 2, \dots, n+1\}$. Actions of the Kashiwara operators \tilde{e}_i, \tilde{f}_i for $i \neq 0$ coincide with the one described in [8], and actions of \tilde{e}_0, \tilde{f}_0 are determined by using the promotion operator [26]. Since we do not use explicit forms of these operators, we omit the details. See [18] for complements of this section.

For two crystals B and B' , one can define the tensor product $B \otimes B' = \{b \otimes b' \mid b \in B, b' \in B'\}$. On the tensor product, actions of the Kashiwara operators have simple form. Namely, the operators \tilde{e}_i, \tilde{f}_i act on $B \otimes B'$ by

$$\begin{aligned}\tilde{e}_i(b \otimes b') &= \begin{cases} \tilde{e}_i b \otimes b' & \text{if } \varphi_i(b) \geq \varepsilon_i(b') \\ b \otimes \tilde{e}_i b' & \text{if } \varphi_i(b) < \varepsilon_i(b'), \end{cases} \\ \tilde{f}_i(b \otimes b') &= \begin{cases} \tilde{f}_i b \otimes b' & \text{if } \varphi_i(b) > \varepsilon_i(b') \\ b \otimes \tilde{f}_i b' & \text{if } \varphi_i(b) \leq \varepsilon_i(b'). \end{cases}\end{aligned}$$

Here we set $\varepsilon_i(b) = \max\{m \geq 0 \mid \tilde{e}_i^m b \neq 0\}$ and $\varphi_i(b) = \max\{m \geq 0 \mid \tilde{f}_i^m b \neq 0\}$. We assume that $0 \otimes b'$ and $b \otimes 0$ as 0. Then it is known that there is a unique crystal isomorphism $R : B^{r,s} \otimes B^{r',s'} \xrightarrow{\sim} B^{r',s'} \otimes B^{r,s}$. We call this map (classical) combinatorial R and usually write the map R simply by \simeq .

Let us consider the affinization of the crystal B . As the set, it is

$$\text{Aff}(B) = \{b[d] \mid b \in B, d \in \mathbb{Z}\}. \quad (1)$$

For the tensor product $b[d] \otimes b'[d'] \in \text{Aff}(B) \otimes \text{Aff}(B')$, we can lift the (classical) combinatorial R to affine case as follows:

$$b[d] \otimes b'[d'] \xrightarrow{R} \tilde{b}'[d' - H(b \otimes b')] \otimes \tilde{b}[d + H(b \otimes b')], \quad (2)$$

where $b \otimes b' \simeq \tilde{b}' \otimes \tilde{b}$ is the isomorphism of (classical) combinatorial R . The function $H(b \otimes b')$ is called the energy function. We will give explicit forms of the combinatorial R and energy function in the subsequent sections. Before closing this subsection, we note the following important property.

Proposition 2.1 *The following Yang–Baxter equation holds on $\text{Aff}(B) \otimes \text{Aff}(B') \otimes \text{Aff}(B'')$:*

$$(R \otimes 1)(1 \otimes R)(R \otimes 1) = (1 \otimes R)(R \otimes 1)(1 \otimes R). \quad (3)$$

2.2 Schensted bumping algorithm

In order to give the explicit form of the combinatorial R and energy function, we recall the Schensted bumping algorithm [22]. Consider the semi-standard Young tableau Y and positive integer x . Then the insertion $(Y \leftarrow x)$ is obtained by the following steps:

1. Insert x in the first row of Y either by displacing the leftmost smallest number which is strictly larger than x , or if no number is larger than x , by adding x on the right of the first row.

2. If x displaced a number from the first row, then insert this number in the second row either by displacing the leftmost smallest number which is strictly larger than it or by adding it on the right of the second row.
3. Repeat this process row by row until some number is added on the right of a row. Here, we assume that there is an empty row on the bottom of Y .

We give an example of this procedure:

$$\left(\begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2 & 2 & 3 & 4 & 5 & 5 \\ \hline 2 & 2 & 3 & 3 & 3 & 4 & 4 & 5 & & \\ \hline 3 & 4 & 5 & 5 & 6 & & & & & \\ \hline \end{array} \leftarrow 2 \right) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2 & 2 & 2^* & 4 & 5 & 5 \\ \hline 2 & 2 & 3 & 3 & 3 & 3^* & 4 & 5 & & \\ \hline 3 & 4 & 4^* & 5 & 6 & & & & & \\ \hline 5^* & & & & & & & & & \\ \hline \end{array}.$$

Here, the numbers with asterisks are the displaced ones. We denote the successive applications of the row insertions by $(Y \leftarrow xy) = ((Y \leftarrow x) \leftarrow y)$ and so on.

We can easily infer the inverse of the bumping procedure $(Y \leftarrow x)$. Namely, starting from the node $(Y \leftarrow x) \setminus Y (= y)$, we do inverse of row insertions until x is ejected from Y . Elementary steps in the inverse procedure are as follows: start from integer y and insert it to row w just above y . Denote by y' the rightmost integer of w that is strictly smaller than y , then new row w' is the row which is obtained by displacing y' by y . Insert y' to the upper row and repeat until the top row.

2.3 Algorithm for combinatorial R and energy function

We give the explicit description of the combinatorial R and the energy function on $B^{r,s} \otimes B^{r',s'}$. We begin with a few terminologies on Young tableaux. Denote rows of a Young tableau Y by y_1, y_2, \dots, y_r from top to bottom. Then row word $row(Y)$ is defined by concatenating rows as $row(Y) = y_r y_{r-1} \dots y_1$. Let $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ be two partitions. Then concatenation of x and y is the partition $(x_1 + y_1, x_2 + y_2, \dots)$.

Proposition 2.2 ([26]) $b \otimes b' \in B^{r,s} \otimes B^{r',s'}$ is mapped to $\tilde{b}' \otimes \tilde{b} \in B^{r',s'} \otimes B^{r,s}$ under the combinatorial R , i.e.,

$$b \otimes b' \xrightarrow{R} \tilde{b}' \otimes \tilde{b}, \quad (4)$$

if and only if

$$(b' \leftarrow row(b)) = (\tilde{b} \leftarrow row(\tilde{b}')). \quad (5)$$

Moreover, the energy function $H(b \otimes b')$ is given by the number of nodes of $(b' \leftarrow row(b))$ outside the concatenation of partitions (s^r) and $(s'^{r'})$.

In order to describe the algorithm for finding \tilde{b} and \tilde{b}' from the data $(b' \leftarrow row(b))$, we introduce a terminology. Let Y be a tableau, and Y' be a subset of Y such that Y' is also a tableau. Consider the set theoretic subtraction $\theta = Y \setminus Y'$. If the number of nodes contained in θ is r , and if the number of nodes of θ contained in each row is always 0 or 1, then θ is called vertical r -strip.

Given a tableau $Y = (b' \leftarrow \text{row}(b))$, let Y' be the upper left part of Y whose shape is (s^r) . We assign numbers from 1 to $r's'$ for each node contained in $\theta = Y \setminus Y'$ by the following procedure. Let θ_1 be the vertical r' -strip of θ as upper as possible. For each node in θ_1 , we assign numbers 1 through r' from bottom to top. Next we consider $\theta \setminus \theta_1$, and find the vertical r' strip θ_2 by the same way. Continue this procedure until all nodes of θ are assigned numbers up to $r's'$. Then we apply inverse bumping procedure according to the labeling of nodes in θ . Denote by u_1 the integer which is ejected when we apply inverse bumping procedure starting from the node with label 1. Denote by Y_1 the tableau such that $(Y_1 \leftarrow u_1) = Y$. Next we apply inverse bumping procedure starting from the node of Y_1 labeled by 2, and obtain the integer u_2 and tableau Y_2 . We do this procedure until we obtain $u_{r's'}$ and $Y_{r's'}$. Finally, we have

$$\tilde{b}' = (\emptyset \leftarrow u_{r's'} u_{r's'-1} \cdots u_1), \quad \tilde{b} = Y_{r's'}. \quad (6)$$

Example 2.3 Consider the following tensor product:

$$b \otimes b' = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 4 \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline 3 & 4 \\ \hline 4 & 5 \\ \hline 5 & 6 \\ \hline \end{array} \in B^{2,2} \otimes B^{3,2}.$$

From b , we have $\text{row}(b) = 2411$, hence we have

$$\left(\begin{array}{|c|c|} \hline 3 & 4 \\ \hline 4 & 5 \\ \hline 5 & 6 \\ \hline \end{array} \leftarrow 2411 \right) = \begin{array}{|c|c|c|} \hline 1 & 1 & 4_3 \\ \hline 2 & 4 & \\ \hline 3_6 & 5_2 & \\ \hline 4_5 & 6_1 & \\ \hline 5_4 & & \\ \hline \end{array}.$$

Here subscripts of each node indicate the order of inverse bumping procedure. For example, we start from the node 6_1 and obtain

$$\left(\begin{array}{|c|c|c|} \hline 1 & 4 & 4 \\ \hline 2 & 5 & \\ \hline 3 & 6 & \\ \hline 4 & & \\ \hline 5 & & \\ \hline \end{array} \leftarrow 1 \right) = \begin{array}{|c|c|c|} \hline 1 & 1 & 4 \\ \hline 2 & 4 & \\ \hline 3 & 5 & \\ \hline 4 & 6 & \\ \hline 5 & & \\ \hline \end{array}, \quad \text{therefore,} \quad Y_1 = \begin{array}{|c|c|c|} \hline 1 & 4 & 4_3 \\ \hline 2 & 5 & \\ \hline 3_6 & 6_2 & \\ \hline 4_5 & & \\ \hline 5_4 & & \\ \hline \end{array}, \quad u_1 = 1.$$

Next we start from the node 6_2 of Y_1 . Continuing in this way, we obtain $u_6 u_5 \cdots u_1 =$

$$321541 \text{ and } Y_6 = \begin{array}{|c|c|} \hline 4 & 4 \\ \hline 5 & 6 \\ \hline \end{array}. \text{ Since } (\emptyset \leftarrow 321541) = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 4 \\ \hline 3 & 5 \\ \hline \end{array}, \text{ we obtain}$$

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 4 \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline 3 & 4 \\ \hline 4 & 5 \\ \hline 5 & 6 \\ \hline \end{array} \simeq \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 4 \\ \hline 3 & 5 \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline 4 & 4 \\ \hline 5 & 6 \\ \hline \end{array}, \quad H \left(\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 4 \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline 3 & 4 \\ \hline 4 & 5 \\ \hline 5 & 6 \\ \hline \end{array} \right) = 3.$$

Note that the energy function is derived from the concatenation of shapes of b and

$$b', \text{ i.e., } \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}.$$

3 Kirillov–Schilling–Shimozono bijection as energy function

In this section, we present a procedure to obtain the image of the Kirillov–Schilling–Shimozono (KSS) bijection by using the energy function and combinatorial R . Necessary facts about the KSS bijection are summarized in Appendix. Let us consider the element $b = b_1 \otimes b_2 \otimes \cdots \otimes b_L \in B^{\alpha_1, \beta_1} \otimes B^{\alpha_2, \beta_2} \otimes \cdots \otimes B^{\alpha_L, \beta_L}$, which we call path. Considering b_j as tableau, we denote each column of it by $b_j = c_{\beta_j} \cdots c_2 c_1$. Then we define the subsets of the tableau $b_{j,k} = c_k \cdots c_2 c_1$. We express the isomorphism of the combinatorial R

$$a \otimes b \simeq b' \otimes a' \quad (7)$$

by the following vertex diagram:

$$\begin{array}{c} b \\ | \\ a \text{---} \text{---} a' \\ | \\ b' \end{array}$$

Successive applications of the combinatorial R are depicted by concatenating these vertices. Let $u_{l,0}^{(a)} = u_l^{(a)} \in B^{a,l}$ be the highest weight element and define $u_{l,j}^{(a)}$ by the following diagram.

$$\begin{array}{ccccccc} & b_1 & & b_2 & & & b_L \\ & | & & | & & & | \\ u_{l,0}^{(a)} & \text{---} & u_{l,1}^{(a)} & \text{---} & u_{l,2}^{(a)} & \cdots \cdots \cdots & u_{l,L-1}^{(a)} & \text{---} & u_{l,L}^{(a)} \\ & | & & | & & & | \\ & b'_1 & & b'_2 & & & b'_L \end{array} \quad (8)$$

We call $u_{l,j}^{(a)}$ *carrier* following terminology used in the box-ball systems. Here the highest weight element $u_l^{(a)} \in B^{a,l}$ is the tableau whose i -th row is occupied by

integers i . For example, $u_4^{(3)} = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 \\ \hline 3 & 3 & 3 & 3 \\ \hline \end{array}$. In the above diagram, let us write

$b' = b'_1 \otimes b'_2 \otimes \cdots \otimes b'_L$. Then we define operator $T_l^{(a)}$ by $T_l^{(a)}(b) = b'$, which is called *time evolution operator of the box-ball systems*.

Associated with each vertex in (8), we define

$$E_{l,j,k}^{(a)} = H \left(u_{l,j-1}^{(a)} \otimes b_{j,k} \right), \quad (l \in \mathbb{Z}_{>0}, 1 \leq j \leq L, 1 \leq k \leq \beta_j). \quad (9)$$

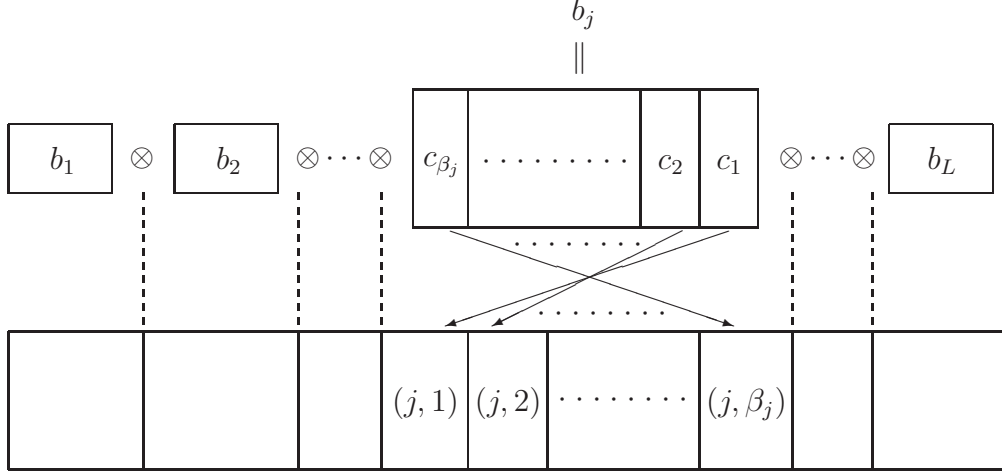
We set $E_{0,j,k}^{(a)} = 0$, $E_{l,j,0}^{(a)} = 0$ and $E_{l,j,k}^{(0)} = 0$.

Definition 3.1 *Local energy distribution (or energy spectrum) is the collection of tables such that the l -th row and the (j, k) -th column of the a -th table ($1 \leq a \leq n$ for $A_n^{(1)}$) is given by*

$$\varepsilon_{l,j,k}^{(a)} := \left(E_{l,j,k}^{(a)} - E_{l,j,k-1}^{(a)} \right) - \left(E_{l-1,j,k}^{(a)} - E_{l-1,j,k-1}^{(a)} \right). \quad (10)$$

Here we assume that the numbering of the columns (j, k) is according to the lexicographic ordering, i.e., $(p, q) < (j, k)$ if $p < j$ and $(j, q) \leq (j, k)$ if $q \leq k$.

Remark 3.2 The meaning of column coordinates (j, k) in the above definition is that j specify the corresponding tensor factor b_j and k specify each column of tableau b_j . We will explain this in more detail by the following diagram.



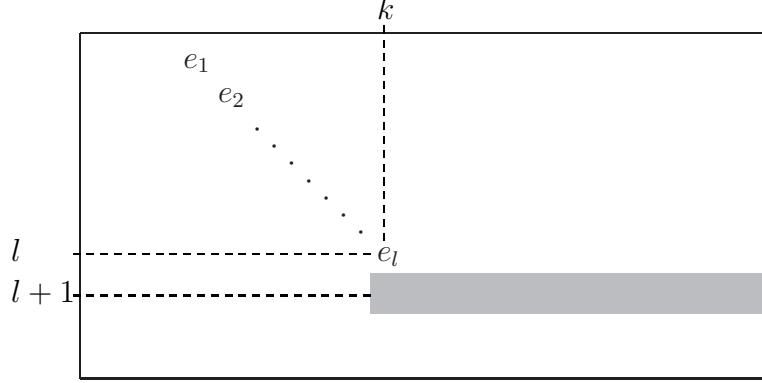
In the above diagram, the first line shows the path $b = b_1 \otimes b_2 \otimes \cdots \otimes b_L$ and the second line shows the corresponding table for local energy distribution. Columns of the local energy distributions are labeled by $(1, 1), \dots, (1, \beta_1), \dots, (L, 1), \dots, (L, \beta_L)$ from left to right. In other words, the local energy distribution is divided into L regions corresponding to b_1, b_2, \dots, b_L from left to right. Now let us look at b_j in more details. Recall that we named columns of tableau b_j by $b_j = c_{\beta_j} \cdots c_2 c_1$, and defined $b_{j,k} = c_k \cdots c_2 c_1$. As we see in Eq.(10), $\varepsilon_{l,j,k}^{(a)}$ is defined by taking difference between k and $k - 1$. In other words, we consider difference between two tableaux $b_{j,k}$ and $b_{j,k-1}$. In this meaning, (j, k) -th column corresponds to the column c_k . Note that c_k is the k -th column of b_j from right, on the other hand, (j, k) is the k -th column of the local energy distribution (to be more specific, sub-region of it corresponding to b_j) from left. In this sense, left and right are reversed within the region of the local energy distribution corresponding to b_j .

Given b , consider the following procedure from Step 1 to 5.

1. Draw the local energy distribution for b . We do the following steps independently with respect to the a -th table ($1 \leq a \leq n$).
2. Starting from the rightmost strictly positive integer of the top row, choose strictly positive integers successively as follows. Assume that we have already chosen the one in the l -th row, k -th column. Take the set of all strictly positive integers of the $(l + 1)$ -th row that are located between the k -th column and the right end of the table.

As an explanation, we give a schematic diagram. In the following diagram, rectangular frame represent the local energy distribution and we denote the

chosen letter at i -th row by e_i . Especially, the letter e_l is at l -th row, k -th column. All other letters are suppressed for simplicity. Then we consider the set of all strictly positive integers contained in the gray strip at $(l+1)$ -th row.



If the set is empty, then stop. Otherwise choose the rightmost element of the set and continue until stop. This procedure enables us to make the rightmost possible successive group consisting of strictly positive integers. For example, in the above diagram, letters on the right of e_i of the i -th row are all 0. Write the position of the lastly picked integer as $\mu_1^{(a)}$ -th row, $(j_1^{(a)}, k_1^{(a)})$ -th column.

3. Continue the previous grouping procedure from right to left. To be more precise, after making a group, we subtract 1 from all letters in the group and get new table. Then we apply Step 2 to this new table again to get the next group. We continue this grouping procedure until all elements of the a -th table become 0. From lower endpoints of thus obtained groups, we obtain the following sequence:

$$\left(\mu_1^{(a)}, (j_1^{(a)}, k_1^{(a)}) \right), \left(\mu_2^{(a)}, (j_2^{(a)}, k_2^{(a)}) \right), \dots, \left(\mu_{N^{(a)}}^{(a)}, (j_{N^{(a)}}^{(a)}, k_{N^{(a)}}^{(a)}) \right). \quad (11)$$

4. Calculate integers $r_1^{(a)}, r_2^{(a)}, \dots, r_{N^{(a)}}^{(a)}$ as follows:

$$r_s^{(a)} = \mathcal{C} + \mathcal{E} \quad (12)$$

$$\mathcal{C} = \sum_{i=1}^{j_s^{(a)}-1} \delta_{\alpha_i, a} \min(\mu_s^{(a)}, \beta_i) + \delta_{\alpha_{j_s^{(a)}}, a} \min(\mu_s^{(a)}, k_s^{(a)}) \quad (13)$$

$$\mathcal{E} = \sum_{(p,q) \leq (j_s^{(a)}, k_s^{(a)})} \sum_{l \leq \mu_s^{(a)}} \left(\varepsilon_{l,p,q}^{(a-1)} - 2\varepsilon_{l,p,q}^{(a)} + \varepsilon_{l,p,q}^{(a+1)} \right) \quad (14)$$

In other words, summation of \mathcal{E} runs over all entries of upper left rectangular region of the local energy distribution whose lower right corner is $\mu_s^{(a)}$ -th row, $(j_s^{(a)}, k_s^{(a)})$ -th column.

5. From the shape of b , we extract the data

$$\mathcal{Q} = \left((\nu_i^{(0)})_{i=1}^{L^{(0)}}, (\nu_i^{(1)})_{i=1}^{L^{(1)}}, \dots, (\nu_i^{(n-1)})_{i=1}^{L^{(n-1)}} \right) \quad (15)$$

as follows. We set $\mathcal{Q} = \emptyset$ as the initial condition. Then, proceeding from the left end of b to right, we add $\nu_i^{(a)}$ to \mathcal{Q} if there is a factor of shape $B^{a+1, \nu_i^{(a)}}$. Finally, we obtain the following data:

$$\left((\nu_i^{(0)})_{i=1}^{L^{(0)}}, \dots, (\nu_i^{(n-1)})_{i=1}^{L^{(n-1)}}, (\mu_i^{(1)}, r_i^{(1)})_{i=1}^{N^{(1)}}, \dots, (\mu_i^{(n)}, r_i^{(n)})_{i=1}^{N^{(n)}} \right). \quad (16)$$

The following is the main result of this paper.

Theorem 3.3 *Given the path $b = b_1 \otimes b_2 \otimes \dots \otimes b_L \in B^{\alpha_1, \beta_1} \otimes B^{\alpha_2, \beta_2} \otimes \dots \otimes B^{\alpha_L, \beta_L}$, we obtain integers $\nu_i^{(a)}$, $(\mu_i^{(a)}, r_i^{(a)})$ according to the above procedure from Step 1 to 5. Then the result coincides with the image of the KSS bijection ϕ :*

$$\phi(b) = \left((\nu_i^{(0)})_{i=1}^{L^{(0)}}, \dots, (\nu_i^{(n-1)})_{i=1}^{L^{(n-1)}}, (\mu_i^{(1)}, r_i^{(1)})_{i=1}^{N^{(1)}}, \dots, (\mu_i^{(n)}, r_i^{(n)})_{i=1}^{N^{(n)}} \right). \quad (17)$$

Remark 3.4 (a) If we start from non-highest weight element b , the above theorem is true if we replace the “rigged configuration” by the “unrestricted rigged configuration” (see the latter part of Section A.3 for explanation). (b) In the local energy distribution, the entries are given by 0 or strictly positive integers (as we will see at the beginning of the proof), and the number of non-zero entries is finite, which will be evaluated in Proposition 4.1.

Remark 3.5 It is known that the time evolution operator $T_l^{(a)}$ can be linearized by the KSS bijection [12]. When $\phi(b)$ is given by

$$\left((\nu_i^{(0)})_{i=1}^{L^{(0)}}, \dots, (\nu_i^{(n-1)})_{i=1}^{L^{(n-1)}}, (\mu_i^{(1)}, r_i^{(1)})_{i=1}^{N^{(1)}}, \dots, (\mu_i^{(n)}, r_i^{(n)})_{i=1}^{N^{(n)}} \right), \quad (18)$$

then $\phi\left(T_l^{(a)}(b)\right)$ is given by

$$\left((\nu_i^{(0)})_{i=1}^{L^{(0)}}, \dots, (\nu_i^{(n-1)})_{i=1}^{L^{(n-1)}}, (\mu_i^{(1)}, r_i^{(1)})_{i=1}^{N^{(1)}}, \dots, (\mu_i^{(a)}, r_i^{(a)} + \min(l, \mu_i^{(a)}))_{i=1}^{N^{(a)}}, \dots, (\mu_i^{(n)}, r_i^{(n)})_{i=1}^{N^{(n)}} \right). \quad (19)$$

In this sense, the groups obtained in Theorem 3.3 represent solitons contained in a path. The proof of this property uses Theorem A.2. For the tensor products of the form $B^{1, s_1} \otimes \dots \otimes B^{1, s_L}$, these solitons actually identified with those arise in the KP hierarchy [15].

Example 3.6 Let us consider the following path:

$$b = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 3 \\ \hline 3 & 4 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 4 \\ \hline 2 & 2 & 3 & 5 \\ \hline \end{array} \quad (20)$$

Note that we give calculation according to the original combinatorial procedure in Example A.1. If we compare Example A.1 with the data given in the following, the reader will find that if we remove a box from l -th column of $\mu^{(a)}$ then there is 1 at l -th row of the corresponding column of a -th table of the local energy distribution (see Remark 3.2 for the correspondence of columns of the path and the local energy distribution).

Now the local energy distribution (Definition 3.1) for b takes the following forms.

$$a = 1 : \quad \begin{array}{|cccc|cc|cccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$
$$a = 2 : \quad \begin{array}{|cccc|cc|cccc} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$
[illegible]
$$a = 4: \quad \begin{array}{|cccc|cc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

We classify the strictly positive integers contained in the above tables as follows:

$a = 1 :$		1	1 1*
-----------	--	---	---------

$a = 2 :$		2^*	1 1 1^*
-----------	--	-------	---------------------

$a = 3 :$		2^*	1 1^*
-----------	--	-------	--------------

$a = 4 :$			1^*
-----------	--	--	-------

Here, according to the grouping procedure in Steps 2 and 3 in Theorem 3.3, labellings of each group are given from right to left, and letters with asterisks show lower end points of each group.

In the $a = 1$ -st table, we recognize one group of cardinality 3 (labeled by 1) whose end point is at 3-rd row, (3, 2)-th column. According to Eq.(13), we have

$$\mathcal{C} = \min(3, 4) + 0 + 0 = 3,$$

where the first term in the middle shows the contribution from $\boxed{1|1|1|1}$ (i.e., $(\alpha_1, \beta_1) = (1, 4)$). According to Eq.(14), we have (the contribution from $\varepsilon_{l,j,k}^{(0)} = 0$)

$$\mathcal{E} = \{0 + 0 + 0 + 0\} + \{(-2 \cdot 1 + 1) + 0\} + \{(-2 \cdot 1 + 2) + (-2 \cdot 1 + 1)\} = -2,$$

hence we have $\mathcal{C} + \mathcal{E} = 1$. Here, the parentheses $\{\cdot\}$ represent tensor factors from the left one to right, and the integers inside represent $\sum_{l=1}^{\mu} \varepsilon_{l,j,k}^{(a)}$ where μ is the cardinality of the corresponding group. Recall that $\sum_{l=1}^{\mu} \varepsilon_{l,j,k}^{(a)}$ is sum of integers within the first μ rows of (j, k) -th column of the local energy distribution.

In the $a = 2$ -nd table, we recognize two groups of cardinalities 3 and 1 (labeled by 1 and 2, respectively). Let us start from the right group whose end point is at 3-rd row, $(3, 2)$ -th column. Then we have

$$\begin{aligned}\mathcal{C} &= 0 + 0 + \min(3, 2) = 2, \\ \mathcal{E} &= \{0 + 0 + 0 + 0\} + \{(1 - 2 \cdot 1 + 1) + 0\} + \{(1 - 2 \cdot 2 + 2) + (1 - 2 \cdot 1 + 0)\} \\ &= -2,\end{aligned}$$

hence we have the rigging $\mathcal{C} + \mathcal{E} = 0$. The left group has end point at 1-st row, $(2, 1)$ -st column, and we have

$$\begin{aligned}\mathcal{C} &= 0 + 0 = 0, \\ \mathcal{E} &= \{0 + 0 + 0 + 0\} + \{(1 - 2 \cdot 1 + 1)\} = 0\end{aligned}$$

hence we have the rigging $\mathcal{C} + \mathcal{E} = 0$.

In the $a = 3$ -rd table, we recognize two groups of cardinalities 2 and 1 (labeled by 1 and 2, respectively). Let us start from the right group whose end point is at 2-nd row, $(3, 1)$ -th column. Then we have

$$\begin{aligned}\mathcal{C} &= 0 + \min(2, 2) + 0 = 2, \\ \mathcal{E} &= \{0 + 0 + 0 + 0\} + \{(1 - 2 \cdot 1 + 0) + 0\} + \{(2 - 2 \cdot 2 + 1)\} = -1,\end{aligned}$$

hence we have the rigging $\mathcal{C} + \mathcal{E} = 0$. Next, we consider the left group whose end point is at 1-st row, $(2, 1)$ -th column. Then we have

$$\begin{aligned}\mathcal{C} &= 0 + \min(3, 1) = 1, \\ \mathcal{E} &= \{0 + 0 + 0 + 0\} + \{(1 - 2 \cdot 1 + 0)\} = -1,\end{aligned}$$

hence we have the rigging $\mathcal{C} + \mathcal{E} = 0$.

Finally, in the $a = 4$ -th table, we recognize one group whose end point is 1-st row, $(3, 1)$ -th column. Then we have

$$\begin{aligned}\mathcal{C} &= 0 + 0 + 0 = 0, \\ \mathcal{E} &= \{0 + 0 + 0 + 0\} + \{1 + 0\} + \{1 - 2 \cdot 1\} = 0,\end{aligned}$$

hence we have the rigging $\mathcal{C} + \mathcal{E} = 0$.

As the result, we obtain the following rigged configuration:

$$\begin{array}{ccccccccc}\nu^{(0)} & & \nu^{(1)} & & \nu^{(2)} & & & & & \\ \boxed{}\boxed{}\boxed{}\boxed{} & & \boxed{}\boxed{}\boxed{}\boxed{} & & \boxed{}\boxed{} & & & & & \\ & & \mu^{(1)} & & \mu^{(2)} & & \mu^{(3)} & & \mu^{(4)} & \\ & & \boxed{}\boxed{}\boxed{} 1 & & \boxed{}\boxed{}\boxed{} 0 & & \boxed{}\boxed{} 0 & & \boxed{} 0 & \\ & & & & \phantom{\boxed{}\boxed{}\boxed{}} 0 & & \phantom{\boxed{}\boxed{}} 0 & & & \end{array}$$

Here we put the riggings on the right of the corresponding rows.

Example 3.7 When the path is not a highest weight element, we can use the same procedure. Consider the following path:

$$b = \begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline 2 & 3 & 3 \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline 2 & 3 & 3 & 3 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \hline 3 & 3 & 4 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 3 & 3 \\ \hline 2 & 3 & 4 & 4 & 4 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 2 & 3 \\ \hline 3 & 3 & 3 & 4 \\ \hline \end{array}$$

Then its local energy distribution takes the following forms.

$$a = 1 :$$

1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$$a = 2 :$$

1	0	0	0	0	0	1	0	0	0	0	0	0	0	2	0	0	0	0	0	0
0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$$a = 3 :$$

0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

We classify the strictly positive integers into groups in the following way:

$$a = 1 :$$

3		2		1	
3*		2			1*
				2	
				2*	

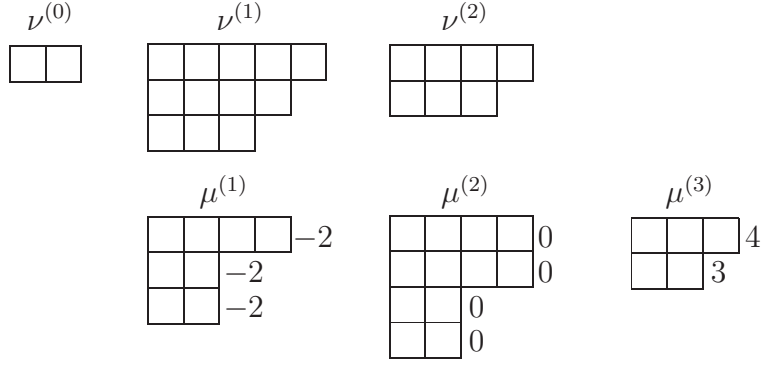
$$a = 2 :$$

4		3		21	
4*		3		2	1*
		3		2	
				3*	2*

$$a = 3 :$$

			2	1	
				2	1*
				2*	

From these data, we obtain the following unrestricted rigged configuration:



4 Energy function and shape of configuration

4.1 Formula

Consider a path $b = b_1 \otimes b_2 \otimes \cdots \otimes b_L \in B^{\alpha_1, \beta_1} \otimes B^{\alpha_2, \beta_2} \otimes \cdots \otimes B^{\alpha_L, \beta_L}$. The goal of this section is to show the following property.

Proposition 4.1 *We have*

$$E_l^{(a)} = Q_l^{(a)}, \quad (21)$$

where we write $E_l^{(a)} = \sum_{j=1}^L E_{l,j,\beta_j}^{(a)}$, and $Q_l^{(a)}$ is defined in Eq.(48).

The meaning of the above quantity $E_l^{(a)}$ is as follows. Let us assume that b is an element of a tensor product of affine crystals. Then from Eq.(2), we have

$$u_l^{(a)}[0] \otimes b \simeq T_l^{(a)}(b) \otimes u^*[E_l^{(a)}]. \quad (22)$$

$E_l^{(a)}$ is a generalization of conserved quantities of the box-ball systems introduced by Fukuda–Okado–Yamada [2]. Their result stems from the Yang–Baxter relation (Proposition 2.1).

4.2 Lemmas about energy functions

We begin with basic lemmas about the energy function. The following is the immediate consequence of the definition in Proposition 2.2.

Lemma 4.2 *Consider the highest elements $u_s^{(r)} \in B^{r,s}$ and $u_{s'}^{(r')} \in B^{r',s'}$ for arbitrary r, s, r', s' . Then we have*

$$u_s^{(r)} \otimes u_{s'}^{(r')} \simeq u_{s'}^{(r')} \otimes u_s^{(r)}, \quad H \left(u_s^{(r)} \otimes u_{s'}^{(r')} \right) = 0. \quad (23)$$

Furthermore, we have the following property.

Lemma 4.3 *For any path b and any $l, k \in \mathbb{Z}_{>0}$, $1 \leq r, a \leq n$ we have*

$$E_l^{(r)} \left(u_k^{(a)} \otimes b \right) = E_l^{(r)} \left(b \otimes u_k^{(a)} \right) = E_l^{(r)}(b). \quad (24)$$

Proof. From Lemma 4.2, we have the equality between the first and the third quantities. Therefore it is enough to show

$$H\left(v \otimes u_k^{(a)}\right) = 0 \quad (25)$$

for arbitrary $v \in B^{r,s}$. Denote the rows of v as v_1, \dots, v_r from top to bottom. Since v is semi-standard, the letters of v_i are greater than or equal to i . Consider the insertion $(u_k^{(a)} \leftarrow v_r)$. Since letters of v_r are greater than or equal to 1, and since $v_r = v_{r,1}v_{r,2} \cdots v_{r,s}$ is a non-decreasing integer sequence, we have

$$\left(u_k^{(a)} \leftarrow v_r\right) = \begin{array}{|c|c|c|c|} \hline 1 & 1 & \cdots & 1 \\ \hline 2 & 2 & \cdots & 2 \\ \hline \cdots & \cdots & \cdots & \cdots \\ \hline a & a & \cdots & a \\ \hline \end{array} \begin{array}{|c|c|c|c|} \hline v_{r,1} & v_{r,2} & \cdots & v_{r,s} \\ \hline \end{array} \quad (26)$$

Assuming that $r > 1$, we insert the row v_{r-1} into this $(u_k^{(a)} \leftarrow v_r)$. Since v is semi-standard, we have $v_{r,1} > v_{r-1,1}$. Therefore, if we insert $v_{r-1,1}$, $v_{r,1}$ is bumped down and come to the next to the row $22 \cdots 2$. Insertion of $v_{r-1,2}$ is quite similar, i.e., $v_{r,2}$ is bumped down and come to the next to the row $22 \cdots 2v_{r,1}$. Continuing in this way, we can show that the resulting tableau takes the following shape. If $r \leq a$, we have

$$\left(u_k^{(a)} \leftarrow \text{row}(v)\right) = \begin{array}{|c|c|c|c|} \hline 1 & 1 & \cdots & 1 \\ \hline 2 & 2 & \cdots & 2 \\ \hline \cdots & \cdots & \cdots & \cdots \\ \hline r & r & r & r \\ \hline \cdots & \cdots & \cdots & \cdots \\ \hline a & a & \cdots & a \\ \hline \end{array} \begin{array}{|c|c|c|c|} \hline v_{1,1} & v_{1,2} & \cdots & v_{1,s} \\ \hline v_{2,1} & v_{2,2} & \cdots & v_{2,s} \\ \hline \cdots & \cdots & \cdots & \cdots \\ \hline v_{r,1} & v_{r,2} & \cdots & v_{r,s} \\ \hline \end{array}, \quad (27)$$

and if $r > a$, we have

$$\left(u_k^{(a)} \leftarrow \text{row}(v)\right) = \begin{array}{|c|c|c|c|} \hline 1 & 1 & \cdots & 1 \\ \hline 2 & 2 & \cdots & 2 \\ \hline \cdots & \cdots & \cdots & \cdots \\ \hline a & a & a & a \\ \hline \cdots & \cdots & \cdots & \cdots \\ \hline v_{r,1} & v_{r,2} & \cdots & v_{r,s} \\ \hline \end{array} \begin{array}{|c|c|c|c|} \hline v_{1,1} & v_{1,2} & \cdots & v_{1,s} \\ \hline v_{2,1} & v_{2,2} & \cdots & v_{2,s} \\ \hline \cdots & \cdots & \cdots & \cdots \\ \hline v_{a,1} & v_{a,2} & \cdots & v_{a,s} \\ \hline \end{array}. \quad (28)$$

In both cases, the shape of $(u_k^{(a)} \leftarrow \text{row}(v))$ coincides with the concatenation of shapes of $u_k^{(a)}$ and v . By Proposition 2.2 the corresponding energy is 0. \blacksquare

Lemma 4.4 Consider the following four elements:

$$v = \begin{array}{|c|c|c|c|} \hline 1 & 1 & \cdots & 1 \\ \hline 2 & 2 & \cdots & 2 \\ \hline \cdots & \cdots & \cdots & \cdots \\ \hline a & a & \cdots & a \\ \hline b_1 & b_2 & \cdots & b_s \\ \hline \end{array}, \quad v' = \begin{array}{|c|c|c|c|} \hline 1 & 1 & \cdots & 1 \\ \hline 2 & 2 & \cdots & 2 \\ \hline \cdots & \cdots & \cdots & \cdots \\ \hline a & a & \cdots & a \\ \hline b'_1 & b'_2 & \cdots & b'_{s'} \\ \hline \end{array}, \quad \tilde{v}' = \begin{array}{|c|c|c|c|} \hline 1 & 1 & \cdots & 1 \\ \hline 2 & 2 & \cdots & 2 \\ \hline \cdots & \cdots & \cdots & \cdots \\ \hline a & a & \cdots & a \\ \hline \tilde{b}'_1 & \tilde{b}'_2 & \cdots & \tilde{b}'_{s'} \\ \hline \end{array}, \quad \tilde{v} = \begin{array}{|c|c|c|c|} \hline 1 & 1 & \cdots & 1 \\ \hline 2 & 2 & \cdots & 2 \\ \hline \cdots & \cdots & \cdots & \cdots \\ \hline a & a & \cdots & a \\ \hline \tilde{b}_1 & \tilde{b}_2 & \cdots & \tilde{b}_s \\ \hline \end{array},$$

where $v, \tilde{v} \in B^{a+1,s}$ and $v', \tilde{v}' \in B^{a+1,s'}$. The upper rows of v, \tilde{v} (resp. v', \tilde{v}') coincide with the highest elements $u_s^{(a)} \in B^{a,s}$ (resp. $u_{s'}^{(a)} \in B^{a,s'}$). The bottom rows satisfy $b_i, b'_i, \tilde{b}_i, \tilde{b}'_i \in \{a+1, a+2\}$. The relationship between $b_i, b'_i, \tilde{b}_i, \tilde{b}'_i$ is given by

$$\boxed{b_1 \mid b_2 \mid \cdots \mid b_s} \otimes \boxed{b'_1 \mid b'_2 \mid \cdots \mid b'_{s'}} \simeq \boxed{\tilde{b}_1 \mid \tilde{b}_2 \mid \cdots \mid \tilde{b}_{s'}} \otimes \boxed{\tilde{b}'_1 \mid \tilde{b}'_2 \mid \cdots \mid \tilde{b}'_{s'}}, \quad (29)$$

under the isomorphism $B^{1,s} \otimes B^{1,s'} \simeq B^{1,s'} \otimes B^{1,s}$.

(1) If $k \neq a+1$, we have

$$u_l^{(k)} \otimes v \simeq v \otimes u_l^{(k)}, \quad H(u_l^{(k)} \otimes v) = 0. \quad (30)$$

(2) We have

$$v \otimes v' \simeq \tilde{v}' \otimes \tilde{v}, \quad H(v \otimes v') = H\left(\boxed{b_1 \mid b_2 \mid \cdots \mid b_s} \otimes \boxed{b'_1 \mid b'_2 \mid \cdots \mid b'_{s'}}\right). \quad (31)$$

In other words, the combinatorial R and energy function for $v \otimes v'$ essentially coincide with the $A_1^{(1)}$ type ones if we replace $a+1$ and $a+2$ of $b_i, b'_i, \tilde{b}_i, \tilde{b}'_i$ with 1 and 2, respectively.

Proof. (1) We use the similar argument given in Lemma 4.3. In order to show the isomorphism of the combinatorial R , one can directly compute $(v \leftarrow \text{row}(u_l^{(k)}))$ and $(u_l^{(k)} \leftarrow \text{row}(v))$ and show that resulting two tableaux coincide.

(2) During the computation of $(v' \leftarrow \text{row}(v))$, we arrive at the following intermediate diagram:

$$\begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & \cdots & 1 & 2 & 2 & \cdots & 2 \\ \hline 2 & 2 & \cdots & 2 & \cdots & \cdots & \cdots & \cdots \\ \hline \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline \cdots & \cdots & \cdots & \cdots & a & a & \cdots & a \\ \hline a & a & \cdots & a & b_1 & b_2 & \cdots & b_s \\ \hline b'_1 & b'_2 & \cdots & b'_{s'} & & & & \\ \hline \end{array}. \quad (32)$$

As the final step of $(v' \leftarrow \text{row}(v))$, we insert the row $11 \cdots 1$ into this diagram. If we insert 1, we see that b_1 is bumped, and we have to calculate the insertion of b_1 into the bottom row $b'_1 b'_2 \cdots b'_{s'}$. Continuing in this way, we see that the calculation essentially coincides with that for $(b'_1 b'_2 \cdots b'_{s'} \leftarrow b_1 b_2 \cdots b_s)$, hence we finish the proof. \blacksquare

4.3 Reduction to the $A_1^{(1)}$ case

We reduce the proof of Proposition 4.1 to the $A_1^{(1)}$ case by invoking analogous trick described in Section 2.7 of [12]. These kinds of arguments are reformulation of color separation scheme of the box-ball systems [27] from point of view of the KSS bijection. Given a path $b = b_1 \otimes b_2 \otimes \cdots \otimes b_L \in B^{\alpha_1, \beta_1} \otimes B^{\alpha_2, \beta_2} \otimes \cdots \otimes B^{\alpha_L, \beta_L}$, let the corresponding rigged configuration be

$$\text{RC} = \left((\nu_i^{(0)})_{i=1}^{L(0)}, \cdots, (\nu_i^{(n-1)})_{i=1}^{L(n-1)}, (\mu_i^{(1)}, r_i^{(1)})_{i=1}^{N(1)}, \cdots, (\mu_i^{(n)}, r_i^{(n)})_{i=1}^{N(n)} \right). \quad (33)$$

We consider the following modification of this RC:

$$\begin{aligned} \text{RC}_+ = & \left((\nu_i^{(0)})_{i=1}^{L^{(0)}} \cup (1^{l^{(0)}+m^{(0)}}), \dots, (\nu_i^{(n-1)})_{i=1}^{L^{(n-1)}} \cup (1^{l^{(n-1)}+m^{(n-1)}}), \right. \\ & \left. (\mu_i^{(1)}, r_i^{(1)} + l^{(0)})_{i=1}^{N^{(1)}}, \dots, (\mu_i^{(n)}, r_i^{(n)} + l^{(n-1)})_{i=1}^{N^{(n)}} \right). \end{aligned} \quad (34)$$

We take $l^{(a)} \gg |\mu^{(a)}|$ ($l^{(0)}$ is arbitrary), and integers $m^{(a)}$ are chosen sufficiently large according to the procedure described in the following.

We apply ϕ^{-1} on this RC_+ in the following two different ways. First, we remove $(1^{m^{(n-1)}}), \dots, (1^{m^{(0)}})$ from the quantum space, and then remove the rest of the quantum space $\nu^{(a)}$ by the same way to obtain b from RC. This is possible since the change of the vacancy number induced by the extra $(1^{l^{(a)}})$ is canceled by the shift of riggings. Finally, we are left with the rigged configuration

$$\left((1^{l^{(0)}}), \dots, (1^{l^{(n-1)}}), (\emptyset, \emptyset), \dots, (\emptyset, \emptyset) \right) \quad (35)$$

and the calculation of ϕ^{-1} on it becomes trivial. Therefore, the resulting path \tilde{b} takes the following form:

$$\tilde{b} = \quad (36)$$

$$\boxed{1}^{\otimes l^{(0)}} \otimes \boxed{\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}}^{\otimes l^{(1)}} \otimes \dots \otimes \boxed{\begin{smallmatrix} 1 \\ 2 \\ \vdots \\ n \end{smallmatrix}}^{\otimes l^{(n-1)}} \otimes b \otimes \boxed{1}^{\otimes m^{(0)}} \otimes \boxed{\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}}^{\otimes m^{(1)}} \otimes \dots \otimes \boxed{\begin{smallmatrix} 1 \\ 2 \\ \vdots \\ n \end{smallmatrix}}^{\otimes m^{(n-1)}}.$$

Next, we remove whole $(\nu_i^{(n-1)})_{i=1}^{L^{(n-1)}} \cup (1^{l^{(n-1)}+m^{(n-1)}})$ before removing other part of the quantum space. The integers $m^{(a)}$ are taken to be large enough such that no rows of $\mu^{(i)}$ ($1 \leq i \leq n-1$) become singular until entire removal of $(\nu_i^{(n-1)}) \cup (1^{l^{(n-1)}+m^{(n-1)}})$. The necessity of the addition $(1^{l^{(n-1)}})$ is clarified from the following observation. Let us compare the actual situation on RC_+ with the following $A_1^{(1)}$ rigged configuration

$$\left((\nu_i^{(n-1)})_{i=1}^{L^{(n-1)}} \cup (1^{l^{(n-1)}+m^{(n-1)}}), (\mu_i^{(n)}, r_i^{(n)})_{i=1}^{N^{(n)}} \right). \quad (37)$$

Then the vacancy numbers for RC_+ are increased by the presence of $\mu^{(n-1)}$, and therefore their increment from Eq.(37) are at most $|\mu^{(n-1)}|$. Recall that in RC_+ , the riggings for $\mu^{(n)}$ are increased by $l^{(n-1)}$, and also that we took $l^{(n-1)} \gg |\mu^{(n-1)}|$. Thus, for each row $\mu_i^{(n)}$ of RC_+ , the value of the corriging (see the end of the first paragraph of Section A.2) is smaller than that for the corresponding row of Eq.(37). Note that the corrigings are always non-negative, and if it is zero, then the row is singular and can be removed by ϕ^{-1} . In other words, smaller corrigings means much easier to be removed. Since all boxes of $(\mu_i^{(n)}, r_i^{(n)})_{i=1}^{N^{(n)}}$ of Eq.(37) are removed with $(\nu_i^{(n-1)})_{i=1}^{L^{(n-1)}} \cup (1^{l^{(n-1)}+m^{(n-1)}})$, we deduce that whole of $(\mu_i^{(n)}, r_i^{(n)} + l^{(n-1)})_{i=1}^{N^{(n)}}$ of RC_+ is removed with $(\nu_i^{(n-1)})_{i=1}^{L^{(n-1)}} \cup (1^{l^{(n-1)}+m^{(n-1)}})$. As the result, we obtain

the path which we denote by $b^{(n)}$, and the rest of the rigged configuration takes the following form:

$$\left((\nu_i^{(0)})_{i=1}^{L^{(0)}} \cup (1^{l^{(0)}+m^{(0)}}), \dots, (\nu_i^{(n-2)})_{i=1}^{L^{(n-2)}} \cup (1^{l^{(n-2)}+m^{(n-2)}}), \right. \quad (38)$$

$$\left. (\mu_i^{(1)}, r_i^{(1)} + l^{(0)})_{i=1}^{N^{(1)}}, \dots, (\mu_i^{(n-1)}, r_i^{(n-1)} + l^{(n-2)})_{i=1}^{N^{(n-1)}} \right). \quad (39)$$

Since $\mu^{(a)}$ ($1 \leq a \leq n-1$) do not become singular during this procedure, each tensor factor of $b^{(n)}$ have special property: as a tableau, the upper $n-1$ rows are equal to the highest element, and the bottom row consists of letters n and $n+1$. In this sense, $b^{(n)}$ is obtained by the $A_1^{(1)}$ like KSS bijection. Next, we remove whole $(\nu_i^{(n-2)})_{i=1}^{L^{(n-2)}} \cup (1^{l^{(n-2)}+m^{(n-2)}})$ in a similar manner. We denote the resulting path $b^{(n-1)}$. We repeat the procedure until whole RC_+ is removed. Then, in general, tensor factors of $b^{(a+1)}$ have the following form:

$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & \cdots & 1 \\ \hline 2 & 2 & \cdots & 2 \\ \hline \cdots & \cdots & \cdots & \cdots \\ \hline a & a & \cdots & a \\ \hline b_1 & b_2 & \cdots & b_s \\ \hline \end{array}, \quad (40)$$

where the bottom row consisting of $b_i \in \{a+1, a+2\}$. Finally, we obtain the path

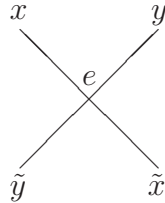
$$\tilde{b} = b^{(1)} \otimes b^{(2)} \otimes \dots \otimes b^{(n)}. \quad (41)$$

Since the difference between these two ways of removal is order of removing rows of the quantum space, these two paths are isomorphic

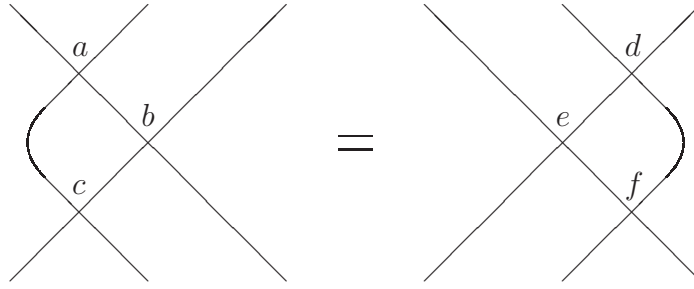
$$\tilde{b} \simeq \tilde{\tilde{b}} \quad (42)$$

due to Theorem A.2.

Proof of Proposition 4.1. From Lemma 4.3, we have $E_l^{(a)}(b) = E_l^{(a)}(\tilde{b})$. Since we have isomorphism $\tilde{b} \simeq \tilde{\tilde{b}}$, we have $E_l^{(a)}(\tilde{b}) = E_l^{(a)}(\tilde{\tilde{b}})$ thanks to the Yang–Baxter relation (Proposition 2.1). To see this, for the isomorphism $x \otimes y \simeq \tilde{y} \otimes \tilde{x}$ with $H(x \otimes y) = e$, let us assign the diagram



Then for tensor product $x \otimes y \otimes z$, the Yang–Baxter relation takes the following expression:



In particular, we can show $a + b = e + f$ (see after Eq.(4.14) of Section 4.3 of [15] for more details). If we put $x = u_l^{(a)}$, we obtain $E_l^{(a)}(y \otimes z) = E_l^{(a)}(\tilde{z} \otimes \tilde{y})$ where $y \otimes z \simeq \tilde{z} \otimes \tilde{y}$. More general case can be deduced inductively from this case.

Now let us evaluate $E_l^{(a)}(\tilde{b})$. From Lemma 4.4 (1), we have $E_l^{(a)}(\tilde{b}) = E_l^{(a)}(b^{(a)})$. Here, we use the following observation. Since integer $m^{(a-1)}$ is large enough, right part of $b^{(a)}$ consists of a large number of highest elements. Therefore, due to Lemma 4.4 (2), the carrier (see Eq.(8) and the comments following it) becomes the highest element $u_l^{(a)}$, hence we can ignore $b^{(a+1)}, \dots, b^{(n)}$. By construction, each $b^{(i)}$ is obtained from RC_+ by ϕ^{-1} on the quantum space $((\nu_k^{(i-1)})_{k=1}^{L^{(i-1)}} \cup (1^{l^{(i-1)}+m^{(i-1)}}))$ and configuration $((\mu_k^{(i)}, r_k^{(i)} + l^{(i-1)})_{k=1}^{N^{(i)}})$. If we compare this with the genuine $A_1^{(1)}$ type KSS bijection on

$$\left((\nu_k^{(i-1)})_{k=1}^{L^{(i-1)}} \cup (1^{l^{(i-1)}+m^{(i-1)}}), (\mu_k^{(i)}, r_k^{(i)} + l^{(i-1)})_{k=1}^{N^{(i)}} \right), \quad (43)$$

the difference is the change of the vacancy numbers due to the presence of other $\mu^{(i)}$. However these changes are constant during construction of $b^{(i)}$. Thus such differences can be absorbed into the change of riggings. In particular, in calculation of $Q_l^{(a)}$, we can ignore such differences. From this observation and Lemma 4.4 (2), we see that if we want to evaluate $E_l^{(a)}(b^{(a)})$, it is enough to consider $A_1^{(1)}$ type rigged configuration (43) with $i = a$.

Therefore, we have to check Proposition 4.1 in $A_1^{(1)}$ case. This can be done by a direct calculation using time evolution of the box-ball systems. Leaving precise arguments for Lemma 4.1 of [20], we explain here main ideas of the proof. Let $b \in B^{1,s_1} \otimes \dots \otimes B^{1,s_L}$ be an arbitrary tensor product of $A_1^{(1)}$ crystals and consider the path $b' := b \otimes \boxed{1}^{\otimes \Lambda}$, where Λ is sufficiently large integer. Note that we have $E_l^{(1)}(b') = E_l^{(1)}(b)$ (see Eq.(25)). Consider time evolution $\bar{b} := \left(T_s^{(1)}\right)^t(b')$ where s and t are sufficiently large integers. From the inverse scattering method for the box-ball systems (see Remark 3.5), we can show that \bar{b} has very simplified structure. This enables us to compute $E_l^{(1)}(\bar{b})$ directly. Again by use of the Yang–Baxter relation (see Theorem 3.2 of [2]), we can show $E_l^{(1)}(\bar{b}) = E_l^{(1)}(b')$. Hence we complete the proof of Proposition 4.1. ■

Remark 4.5 Part of this section is based on joint work with A. Kuniba (April 2007, unpublished).

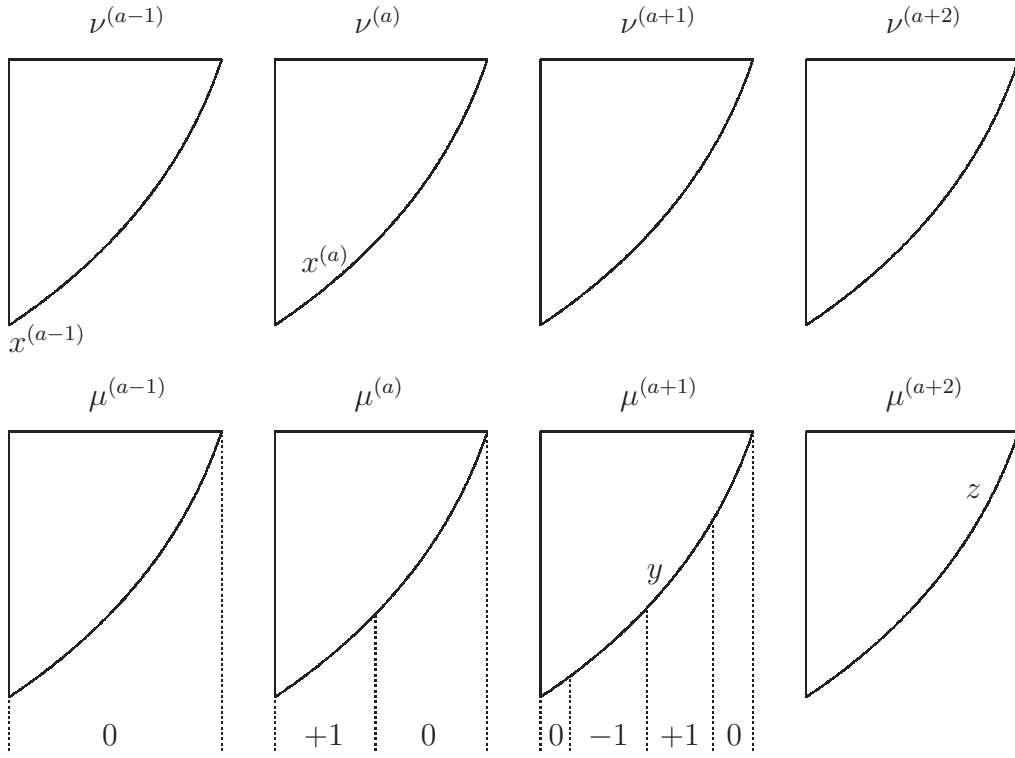


Figure 1: Calculation of ϕ^{-1} : remove $x^{(a)}$, y , z , \dots , and add $x^{(a-1)}$.

5 Proof of Theorem 3.3

To begin with, we have to settle some combinatorial property concerning the calculation of ϕ^{-1} . Given a path, we regard each semi-standard Young tableau (i.e., tensor factor of the path) as collection of columns. We shall examine in details the construction of these columns from the given rigged configuration. Recall that ϕ^{-1} recursively constructs tensor factors of the path from right to left, and within each tensor factor, it recursively constructs columns of tableau from left to right. Therefore, without loss of generality, we look at the leftmost column of the rightmost tensor factor of the path. We denote this column by C , and denote the number of nodes of C by $a + 1$. Then the construction of C starts by removing a box from $\nu^{(a)}$ (we name the box $x^{(a)}$). In the box removing procedure of ϕ^{-1} , starting from the box $x^{(a)}$, we remove other boxes (say $y \in \mu^{(a+1)}$, $z \in \mu^{(a+2)}$, and so on) simultaneously. Then we add a box to the first column of $\nu^{(a-1)}$, which we call $x^{(a-1)}$. The next step starts from $x^{(a-1)}$ and do the similar procedure.

Let us summarize the situation in Figure 1. Here, integers below the diagrams show change of the vacancy numbers caused by the removal of $x^{(a)}$, y , z , \dots , and the addition of $x^{(a-1)}$. We can determine these changes by comparing number of boxes corresponding to each term of definition Eq.(49) before and after the operation in the previous sentence. Let us denote the column coordinate of a box w in $\mu^{(i)}$ (or $\nu^{(i)}$) by $\text{col}(w)$. Then the vacancy number for each row of $\mu^{(a)}$ whose rightmost box w satisfies

$0 \leq \text{col}(w) < \text{col}(y)$ increases by 1, and those for the other region do not change. In the same way, the vacancy numbers for the region $\text{col}(x^{(a)}) \leq \text{col}(w) < \text{col}(y)$ of $\mu^{(a+1)}$ decrease by 1, those for the region $\text{col}(y) \leq \text{col}(w) < \text{col}(z)$ are increased by 1, and those for the other region do not change.

In the following lemma, we will use these settings such as the column C is the leftmost column of the rightmost tensor factor of the path. In other words, the column C is the first column constructed by the procedure ϕ^{-1} .

Lemma 5.1 *Let us exclusively consider the procedure ϕ^{-1} during construction of the column C . Assume that boxes of some row, say μ^i of $\mu^{(i)}$ ($1 \leq i \leq n$) are removed during construction of the column C . Then the vacancy number for the row μ^i evaluated when its rightmost box (i.e., the firstly removed box of the row) is removed is equal to the one evaluated before the construction of C .*

Proof. Construction of the column C under ϕ^{-1} begins with removal of the box $x^{(a)}$. Assume that we have removed boxes $x^{(a)}$, y , z , \dots and added $x^{(a-1)}$. See Figure 1, especially change of the vacancy numbers summarized there. Let us examine which boxes can be removed from $\mu^{(a)}$, $\mu^{(a+1)}$, \dots if we remove $x^{(a-1)}$. First we consider $\mu^{(a)}$. In $\mu^{(a)}$, there is no singular row within the region whose vacancy numbers are increased by removal of $x^{(a)}$, y , z , \dots and addition of $x^{(a-1)}$. Therefore rows w of $\mu^{(a)}$ that satisfy $\text{col}(y) \leq \text{col}(w)$ are the candidate for removal. We see that the vacancy numbers for the rows w of $\mu^{(a)}$ satisfying $\text{col}(y) \leq \text{col}(w)$ do not change from those evaluated before removing $x^{(a)}$.

Assume that we can remove some row of $\mu^{(a)}$, then the next removal, if possible, will be a row of $\mu^{(a+1)}$. Recall that by definition of ϕ^{-1} , column coordinates of simultaneously removed boxes are non decreasing integer sequence (from left to right). Thus we see that the next candidates for removal are within the region of $\mu^{(a+1)}$ whose column coordinates are greater than or equal to $\text{col}(y)$. Here we used the fact that we have removed a box of $\mu^{(a)}$ whose column coordinate is greater than or equal to $\text{col}(y)$ (see the end of the previous paragraph). Note that the row to which the box y was originally belonged is now shorter than $\text{col}(y)$ since its length is shortened by 1 due to the removal of the box y . Note also that the vacancy numbers for rows w of $\mu^{(a+1)}$ that satisfy $\text{col}(y) \leq \text{col}(w) < \text{col}(z)$ are increased by 1 by removal of $x^{(a)}$, y , z , \dots and addition of $x^{(a-1)}$. Therefore we see that candidates for removal exist in the region of $\mu^{(a+1)}$ whose column coordinates are greater than or equal to $\text{col}(z)$. In conclusion, the vacancy number for the row of $\mu^{(a+1)}$ that can be removed simultaneously with $x^{(a-1)}$ remains the same as that evaluated before removing $x^{(a)}$.

We can continue the similar arguments for $\mu^{(a+2)}$, $\mu^{(a+3)}$, \dots . To state the result for general case, let us consider $\mu^{(A)}$ for some $A > a + 1$. We divide the argument into the following two cases.

1. Assume that some row of $\mu^{(A+1)}$ was removed with $x^{(a)}$. Let us call this removed box of $\mu^{(A+1)}$ by X . Then the situation essentially coincides with the

case of $\mu^{(a+1)}$ in place of $\mu^{(A)}$. Namely, if we remove $x^{(a-1)}$, we can simultaneously remove, if possible, a row of $\mu^{(A)}$ whose column coordinate is greater than or equal to $\text{col}(X)$. In such case, the vacancy number for the row of $\mu^{(A)}$ that is removed with $x^{(a-1)}$ is the same as that evaluated before removing $x^{(a)}$.

2. On the other hand, if boxes of $\mu^{(A+1)}$ were not removed with $x^{(a)}$, then we see that the boxes of $\mu^{(A)}$ will not be removed with $x^{(a-1)}$. To see this, let A be the minimal integer satisfying $A \geq a$ such that the boxes of $\mu^{(A+1)}$ were not removed with $x^{(a)}$. If $A = a$, then we remove only one box $x^{(a)}$ from $\nu^{(a)}$ and add a box $x^{(a-1)}$ at the first column of $\nu^{(a-1)}$. Thus the vacancy numbers for all rows of $\mu^{(a)}$ are increased by 1. Therefore we can not remove a box from $\mu^{(a)}$ simultaneously with removal of $x^{(a-1)}$. In the following, we assume that $A > a$. Let us denote the removed box of $\mu^{(A)}$ by X .

- (a) After removing $x^{(a)}$, y , z , \dots , X , the vacancy numbers for the rows of $\mu^{(A)}$ longer than or equal to $\text{col}(X)$ are increased by 1. Therefore we can not remove rows of $\mu^{(A)}$ longer than or equal to $\text{col}(X)$.
- (b) On the other hand, we can apply the above case 1 to $\mu^{(A-1)}$ to see that, if we remove a row of $\mu^{(A-1)}$ with $x^{(a-1)}$, then the removed row is longer than or equal to $\text{col}(X)$. Therefore if we remove a box from $\mu^{(A)}$ with $x^{(a-1)}$, we have to remove a row that is longer than or equal to $\text{col}(X)$. Recall that by definition of ϕ^{-1} , column coordinates of simultaneously removed boxes has to be non-decreasing sequence from left to right.

Combining these two mutually contradicting conditions, we conclude that we can not remove a box from $\mu^{(A)}$ with $x^{(a-1)}$. This is nothing but the column strict semi-standard property of the column C .

To summarize, the vacancy numbers for all rows that are removed simultaneously with $x^{(a-1)}$ are the same with those evaluated before we remove $x^{(a)}$.

More general case corresponding to all $x^{(i)}$ can be shown inductively. Namely, we replace $x^{(a)}$ and $x^{(a-1)}$ with $x^{(i)}$ and $x^{(i-1)}$, respectively, to show the statement for $x^{(i-1)}$ case from the assumption for $x^{(i)}$. ■

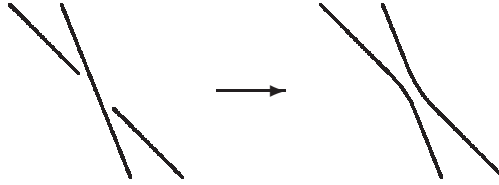
In order to connect the original procedure of ϕ^{-1} with the local energy distribution, we introduce a table which record the process of ϕ^{-1} . More precisely, we prepare array of n tables whose a -th table records the history of $\mu^{(a)}$ as follows. Recall that in the calculation of ϕ^{-1} , we remove a box from the quantum space, say $\nu^{(i)}$ and, at the same time, remove several boxes from $\mu^{(j)}$'s. Then we add a box to $\nu^{(i-1)}$ and do the similar procedure. We do these procedures for several times (in this case, $i+1$ times), then we obtain one column of a semi-standard Young tableau of path. Every time we add a column (say C) to tableau of the path, we add one column to the a -th table (from right to left according to ϕ^{-1}). The column in this a -th table contains the following information: if we remove total of m boxes at k -th

column of $\mu^{(a)}$ during the construction of C , we put m at the k -th row of the column in the a -th table. Furthermore, we draw curves on the table which join entries of the table row by row if they belong to the same row of $\mu^{(a)}$. Thus if an entry is m , then there are m curves run through the position of m .

As a result, we obtain a table whose entries are joined by curves. Then, in general, these curves have crossing with each other:



Here parities of crossings (i.e., which line crosses over the other line) are not important. Note that by construction of the curves joining entries of table, these curves always go rightwards and/or downwards, and never go leftwards and/or upwards when we start from the top end. In order to detect each lower end point \bullet , we start from the rightmost strictly positive integer in the top row. We always go to weakly right of lower rows (do not go to left), and choose positive integers row by row. If we encounter the crossing, we resolve it as follows:



In other words, if we have more than two non-zero positive integers in the lower row which are located weakly right to the chosen integer of the upper row, then we always choose the rightmost one, and proceed. In this way, we arrive at one of the end points. We subtract 1 from the chosen positive integers, and choose the next group. Continuing in this way, we can always detect all the end points.

Proof of Theorem 3.3. First, let us show the equivalence between the local energy distribution and the above table recording the process of ϕ^{-1} . From Proposition 4.1, we give interpretation of differences of $E_{l,j,k}^{(a)}$ of Eq.(9) as follows. As before, consider the path $b = b_1 \otimes b_2 \otimes \cdots \otimes b_L \in B^{\alpha_1, \beta_1} \otimes B^{\alpha_2, \beta_2} \otimes \cdots \otimes B^{\alpha_L, \beta_L}$, and denote the columns of tableau b_j as $b_j = c_{\beta_j} \cdots c_2 c_1$, and set $b_{j,k} = c_k \cdots c_2 c_1$. Consider the path $b_{[k]} = b_1 \otimes b_2 \otimes \cdots \otimes b_{j-1} \otimes b_{j,k}$ and apply Proposition 4.1. Then we see that

$$\sum_{i=1}^{j-1} E_{l,i,\beta_i}^{(a)} + E_{l,j,k}^{(a)} = (Q_l^{(a)} \text{ for the path } b_{[k]}). \quad (44)$$

Therefore we have

$$E_{l,j,k}^{(a)} - E_{l,j,k-1}^{(a)} = (Q_l^{(a)} \text{ for the path } b_{[k]}) - (Q_l^{(a)} \text{ for the path } b_{[k-1]}). \quad (45)$$

By comparing with the procedure of ϕ^{-1} , the meaning for the right hand side is clear. Namely, it represents the number of boxes removed from the first l columns of $\mu^{(a)}$ during construction of the column c_k under ϕ^{-1} . Note that $Q_l^{(a)}$ represents, by definition, the number of boxes contained in the first l columns of $\mu^{(a)}$. Therefore taking difference once more,

$$\left(E_{l,j,k}^{(a)} - E_{l,j,k-1}^{(a)}\right) - \left(E_{l-1,j,k}^{(a)} - E_{l-1,j,k-1}^{(a)}\right), \quad (46)$$

it represents the number of boxes removed from l -th column of $\mu^{(a)}$ corresponding to the column c_k of a path under ϕ^{-1} . Therefore, the table introduced just before the proof is equivalent to the local energy distribution. From this fact, it also follows that entries of the local energy distribution are non-negative. Furthermore, the end points detected in the arguments before the proof correspond to the rightmost box of each row of partitions $\mu^{(a)}$. This follows from the fact that process of ϕ^{-1} proceeds from right to left in the table. Hence we see that Steps 2 and 3 of the procedure given in Theorem 3.3 certainly extract the data $\mu^{(a)}$ ($1 \leq a \leq n$).

Finally, it remains to show that Eq.(12) to Eq.(14) in Step 4 really give the riggings. Consider the arbitrary end point appearing in the local energy distribution, say at $\mu_i^{(a)}$ -th row and (j, k) -th column. As before, consider the path $b_1 \otimes b_2 \otimes \cdots \otimes b_{j,k}$. Then the end point we are considering means that the corresponding row $\mu_i^{(a)}$ is removed for the first time when we are constructing the column c_k of the factor b_j under ϕ^{-1} . By definition of ϕ^{-1} , the rigging is equal to the corresponding vacancy number when the row $\mu_i^{(a)}$ is removed, and by Lemma 5.1, it is enough to evaluate the vacancy number at the time when we begin to construct c_k . This means that it is enough to consider the path $b_1 \otimes b_2 \otimes \cdots \otimes b_{j,k}$, and evaluate the vacancy number for the corresponding rigged configuration. Then \mathcal{C} of Eq.(13) counts the contribution from the quantum space (the first term of the right hand side of Eq.(49)). $\delta_{\alpha_i, a}$ in the expression Eq.(13) reflects the fact that only the contribution from $\nu^{(a-1)}$ appear in the vacancy number, and the second term reflects the fact that we are considering the truncated path $b_1 \otimes b_2 \otimes \cdots \otimes b_{j,k}$. The \mathcal{E} of Eq.(14) counts the contribution from $\mu^{(a-1)}$, $\mu^{(a)}$ and $\mu^{(a+1)}$ of the truncated path (the last three terms of the right hand side of Eq.(49)).

We see that Steps 1 to 5 appearing in the procedure described by our theorem permit to recover the rigged configuration we are considering. This proves the theorem. \blacksquare

A Kirillov–Schilling–Shimozono bijection

A.1 Rigged configurations

In this appendix, we collect necessary facts about the Kirillov–Schilling–Shimozono (KSS) bijection [11]. The reader should consult review [24] for more details. The KSS bijection gives one to one correspondences between elements of tensor products $B^{r_1, s_1} \otimes B^{r_2, s_2} \otimes \dots \otimes B^{r_L, s_L}$ (which we call *paths*) and combinatorial objects called the rigged configurations. The original theory deals with the highest weight elements, and we first consider such situation. Let us define the $A_n^{(1)}$ rigged configurations. Consider the following set of data:

$$\text{RC} = \left((\nu_i^{(0)})_{i=1}^{L^{(0)}}, \dots, (\nu_i^{(n-1)})_{i=1}^{L^{(n-1)}}, (\mu_i^{(1)}, r_i^{(1)})_{i=1}^{N^{(1)}}, \dots, (\mu_i^{(n)}, r_i^{(n)})_{i=1}^{N^{(n)}} \right). \quad (47)$$

Here, $\nu_i^{(a)}$ and $\mu_i^{(a)}$ are positive integer sequences. Now we describe the conditions imposed on this RC. Denote the number of boxes contained in the first l columns of the Young diagrammatic expression of $\mu^{(a)}$ by $Q_l^{(a)}$, i.e.,

$$Q_l^{(a)} = \sum_{i=1}^{N^{(a)}} \min(l, \mu_i^{(a)}). \quad (48)$$

Then, corresponding to the length l row of $\mu^{(a)}$, we define the *vacancy number* by the following formula:

$$p_l^{(a)} = \sum_{i=1}^{L^{(a-1)}} \min(l, \nu_i^{(a-1)}) + Q_l^{(a-1)} - 2Q_l^{(a)} + Q_l^{(a+1)}, \quad (49)$$

where the first term in the right hand side gives the number of boxes contained in the first l columns of $\nu^{(a-1)}$, and we set $Q_l^{(0)} = Q_l^{(n+1)} = 0$. The integer $r_i^{(a)}$ associated with $\mu_i^{(a)}$ is called the *rigging associated with $\mu_i^{(a)}$* if it satisfies the condition

$$0 \leq r_i^{(a)} \leq p_{\mu_i^{(a)}}^{(a)}. \quad (50)$$

Then the above RC is called the *rigged configuration* if all the vacancy numbers are non-negative:

$$0 \leq p_{\mu_i^{(a)}}^{(a)}, \quad (1 \leq a \leq n, 1 \leq i \leq N^{(a)}), \quad (51)$$

and if all integers $r_i^{(a)}$ are the riggings associated with $\mu_i^{(a)}$. We usually call $\nu^{(a)}$ of RC the *quantum space*, and $\mu^{(a)}$ of RC the *configuration*.

A.2 Combinatorial procedures for KSS bijection

By the KSS bijection, each path b is mapped to the rigged configuration RC:

$$\phi : b \longmapsto \text{RC}. \quad (52)$$

Below we describe how to calculate ϕ^{-1} rather than ϕ . The calculation of ϕ^{-1} consisting of array of recursive procedures, which are described by box removing processes. In order to describe which box of $\mu^{(a)}$ to be removed, we use the following notion: if the row $\mu_i^{(a)}$ satisfies the condition

$$p_{\mu_i^{(a)}}^{(a)} = r_i^{(a)}, \quad (53)$$

i.e., the rigging and the corresponding vacancy number coincide, then the row $\mu_i^{(a)}$ is called *singular*. The value $p_{\mu_i^{(a)}}^{(a)} - r_i^{(a)}$ is called *corriging*.

The calculation of ϕ^{-1} starts by choosing which row of $\nu^{(a)}$ ($0 \leq a \leq n-1$) we are going to remove. Hence the map ϕ^{-1} is a function of such choice (see Theorem A.2 below). We choose row, say $\nu_i^{(a)}$, to be removed. From the row $\nu_i^{(a)}$, we obtain an element of the crystal $B^{a+1, \nu_i^{(a)}}$ by the following procedure.

1. Let us denote the rightmost box of the row $\nu_i^{(a)}$ by $x^{(a)}$ (see Figure 1 in the main text). Starting from $x^{(a)}$, we choose rows of $\mu^{(a+1)}$, $\mu^{(a+2)}$, \dots , by the following recursive rule. Assume that we have chosen a row of $\mu^{(j)}$. Then we choose a row of $\mu^{(j+1)}$ which is the shortest singular row among the rows of $\mu^{(j+1)}$ not shorter than the chosen row of $\mu^{(j)}$, and proceed to $\mu^{(j+2)}$. If there are more than one such row, choose one arbitrarily. If there is not such a row, then stop. If we can choose a row from $\mu^{(j_{a-1})}$, and cannot choose in $\mu^{(j_a)}$, we obtain integer j_a as the output.
2. We remove all the rightmost boxes of chosen rows simultaneously, and add a length 1 row to $\nu^{(a-1)}$ which we call $x^{(a-1)}$. After removal and addition, we calculate the new vacancy numbers. New riggings are determined as follows. If the corresponding row is not removed, choose the same rigging as before. On the other hand, if the corresponding row is removed, then choose the new rigging equal to the new vacancy number for the corresponding row.
3. Repeat the above Step 1 and Step 2 starting from $x^{(a-1)}$, and obtain j_{a-1} as an output. We do this until $x^{(0)}$ is removed. Then we put j_a, \dots, j_1, j_0 to the leftmost empty column of $(a+1) \times \nu_i^{(a)}$ Young diagram as follows:

$$\begin{array}{|c|c|c|c|c|c|}
 \hline
 j_0 & & & & & \\
 \hline
 j_1 & & & & & \\
 \hline
 \vdots & & & & & \\
 \hline
 j_a & & & & & \\
 \hline
 \end{array} \quad (54)$$

4. Repeat the above Step 1 through Step 3 for the rest of $\nu_i^{(a)}$ box by box (from right to left) until entire row is removed. Then we obtain an element of the crystal $B^{a+1, \nu_i^{(a)}}$.

We repeat the above Step 1 through Step 4 by choosing rows of the quantum space arbitrarily until all boxes of the quantum space are removed. We take tensor products of these tableaux from right to left.

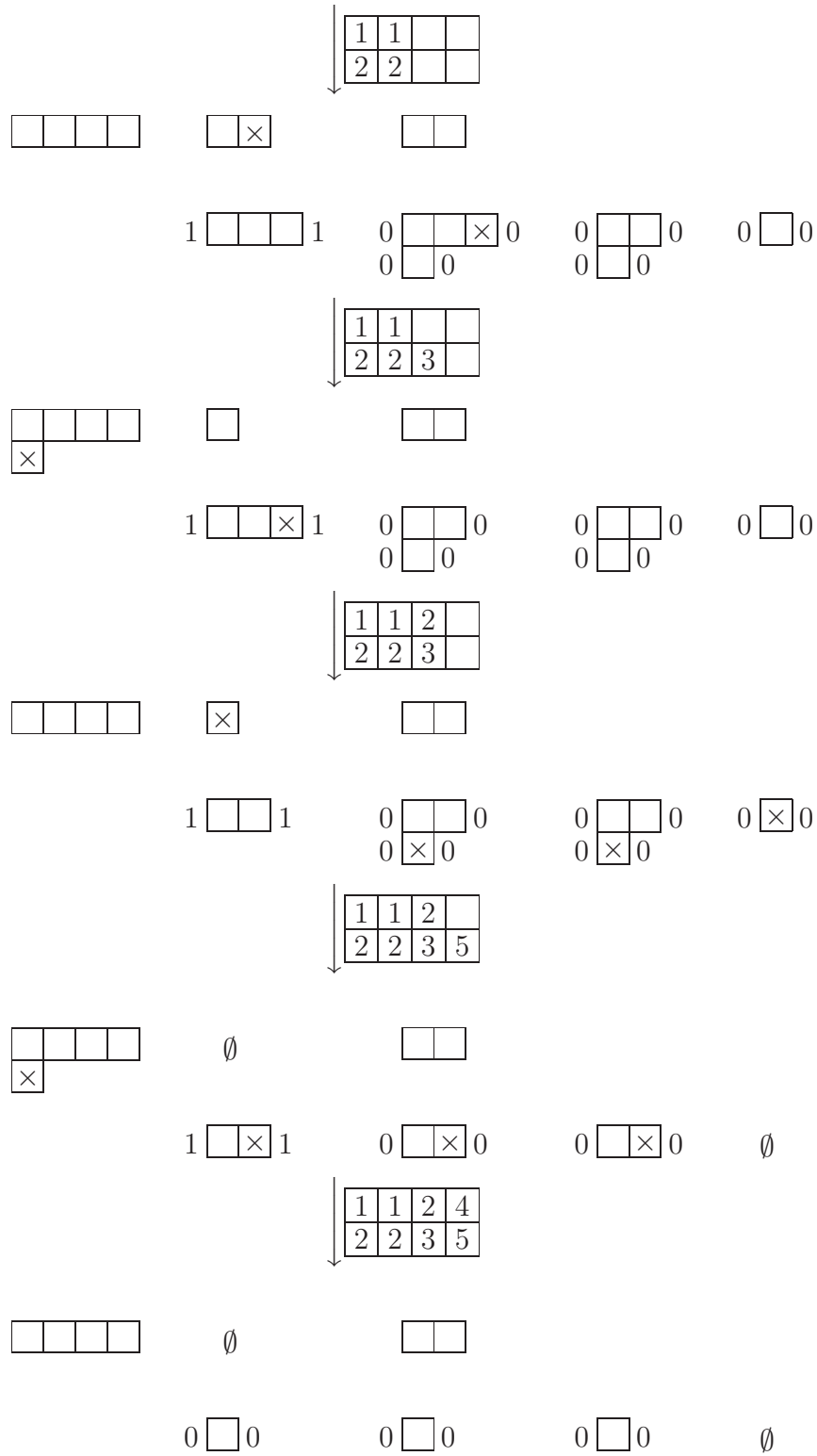
It is known that the above procedure is well-defined, and the inverse map ϕ admits similar description [11]. In fact, the procedure for ϕ is obtained by just reversing the above procedure.

Example A.1 Consider the following rigged configuration:

$$\begin{array}{ccccc}
 \nu^{(0)} & \nu^{(1)} & \nu^{(2)} & & \\
 \boxed{}\boxed{}\boxed{}\boxed{} & \boxed{}\boxed{}\boxed{}\boxed{} & \boxed{}\boxed{} & & \\
 & \mu^{(1)} & \mu^{(2)} & \mu^{(3)} & \mu^{(4)} \\
 & 1\boxed{}\boxed{}\boxed{}1 & 1\boxed{}\boxed{}\boxed{}0 & 0\boxed{}\boxed{}0 & 0\boxed{}0 \\
 & & 0\boxed{}0 & 0\boxed{}0 &
 \end{array}$$

Here we put the riggings (resp. vacancy numbers) on the right (resp. left) of the corresponding rows. As the first step, let us remove $\nu^{(1)}$. Then calculation goes as follows (we put \times to the box to be removed):

$$\begin{array}{ccccc}
 \boxed{}\boxed{}\boxed{}\boxed{} & \boxed{}\boxed{}\boxed{}\boxed{}\times & \boxed{}\boxed{} & & \\
 & 1\boxed{}\boxed{}\boxed{}1 & 1\boxed{}\boxed{}\boxed{}0 & 0\boxed{}\boxed{}0 & 0\boxed{}0 \\
 & & 0\boxed{}0 & 0\boxed{}0 & \\
 & \downarrow \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline 2 & & & & \\ \hline \end{array} & & & \\
 \boxed{}\boxed{}\boxed{}\boxed{} & \boxed{}\boxed{}\boxed{} & \boxed{}\boxed{} & & \\
 \times\boxed{} & & & & \\
 & 2\boxed{}\boxed{}\boxed{}1 & 1\boxed{}\boxed{}\boxed{}0 & 0\boxed{}\boxed{}0 & 0\boxed{}0 \\
 & & 0\boxed{}0 & 0\boxed{}0 & \\
 & \downarrow \begin{array}{|c|c|c|c|c|} \hline 1 & & & & \\ \hline 2 & & & & \\ \hline \end{array} & & & \\
 \boxed{}\boxed{}\boxed{}\boxed{} & \boxed{}\boxed{}\times & \boxed{}\boxed{} & & \\
 & 1\boxed{}\boxed{}\boxed{}1 & 1\boxed{}\boxed{}\boxed{}0 & 0\boxed{}\boxed{}0 & 0\boxed{}0 \\
 & & 0\boxed{}0 & 0\boxed{}0 & \\
 & \downarrow \begin{array}{|c|c|c|c|c|} \hline 1 & & & & \\ \hline 2 & 2 & & & \\ \hline \end{array} & & & \\
 \boxed{}\boxed{}\boxed{}\boxed{} & \boxed{}\boxed{} & \boxed{}\boxed{} & & \\
 \times\boxed{} & & & & \\
 & 2\boxed{}\boxed{}\boxed{}1 & 0\boxed{}\boxed{}\boxed{}0 & 0\boxed{}\boxed{}0 & 0\boxed{}0 \\
 & & 0\boxed{}0 & 0\boxed{}0 &
 \end{array}$$



Continuing in this way, we obtain the following path:

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 3 \\ \hline 3 & 4 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 4 \\ \hline 2 & 2 & 3 & 5 \\ \hline \end{array}.$$

In order to make comparison, we consider the same example in Example 3.6 in the main text.

A.3 Several facts about KSS bijection

As we have seen before, the map ϕ^{-1} is a function of choice of rows of the quantum space. Its dependence is described by the following theorem.

Theorem A.2 ([11], Lemma 8.5) *Take two rows from the quantum space of the rigged configuration arbitrarily, and denote them by ν_a and ν_b . When we remove successively these two rows as ν_a at first and next ν_b under ϕ^{-1} , then we obtain two tableaux, which we denote by a_1 and b_1 , respectively. Next, on the contrary, we first remove ν_b and second ν_a (keeping the order of other removal invariant) and we get b_2 and a_2 . Then we have*

$$b_1 \otimes a_1 \simeq a_2 \otimes b_2, \quad (55)$$

under the isomorphism of the combinatorial R .

Recently, the original KSS bijection was extended to include non-highest weight elements [23, 1]. The combinatorial procedures for ϕ and ϕ^{-1} are the formal extension of the original ones. For arbitrary element $b \in B^{r_1, s_1} \otimes B^{r_2, s_2} \otimes \cdots \otimes B^{r_L, s_L}$, we obtain $\phi(b)$ as an extension of the rigged configuration, which we call *unrestricted rigged configurations*. Characterization for the set of the unrestricted rigged configurations is given in these papers, however we omit it since we do not use it. For our purpose, it is enough to start from arbitrarily given path, and use the prescription given in Section 7.1 of [15]. In the prescription given there, we put suitable prefix to the path, and the effect of such modification of path on the level of the rigged configuration is explicitly estimated. As the result, many apparatus (like Theorem A.2) holds for the case of the unrestricted rigged configuration. In particular, our procedure given in Theorem 3.3 does not depend on whether the path is highest or non-highest.

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References

- [1] L. Deka and A. Schilling, New fermionic formula for unrestricted Kostka polynomials, J. Comb. Theor. Ser. A, **113** (2006) 1435–1461, arXiv:math.CO/0509194.
- [2] K. Fukuda, M. Okado and Y. Yamada, Energy functions in box-ball systems, Int. J. Mod. Phys. **A15** (2000) 1379–1392, arXiv:math.QA/9908116.
- [3] G. Hatayama, K. Hikami, R. Inoue, A. Kuniba, T. Takagi and T. Tokihiro, The $A_M^{(1)}$ automata related to crystals of symmetric tensors, J. Math. Phys. **42** (2001) 274–308, arXiv:math.QA/9912209.

- [4] R. Hirota, *The direct method in soliton theory*, Cambridge Univ. Press (2004).
- [5] M. Jimbo and T. Miwa, Solitons and infinite dimensional Lie algebras, Publ. RIMS. Kyoto Univ. **19** (1983) 943–1001.
- [6] S.-J. Kang, M. Kashiwara, K. C. Misra, T. Miwa, T. Nakashima and A. Nakayashiki, Perfect crystals of quantum affine Lie algebras, Duke Math. J. **68** (1992) 499–607.
- [7] M. Kashiwara, On crystal bases of the q -analogue of universal enveloping algebras, Duke Math. J. **63** (1991) 465–516.
- [8] M. Kashiwara and T. Nakashima, Crystal graphs for representations of the q -analogue of classical Lie algebras, J. Algebra **165** (1994) 295–345.
- [9] S. V. Kerov, A. N. Kirillov and N. Yu. Reshetikhin, Combinatorics, the Bethe ansatz and representations of the symmetric group, J. Soviet Math. **41** (1988) 916–924.
- [10] A. N. Kirillov and N. Yu. Reshetikhin, The Bethe ansatz and the combinatorics of Young tableaux. J. Soviet Math. **41** (1988) 925–955.
- [11] A. N. Kirillov, A. Schilling and M. Shimozono, A bijection between Littlewood–Richardson tableaux and rigged configurations, Selecta Math. (N.S.) **8** (2002) 67–135, arXiv:math.CO/9901037.
- [12] A. Kuniba, M. Okado, R. Sakamoto, T. Takagi and Y. Yamada, Crystal interpretation of Kerov–Kirillov–Reshetikhin bijection, Nucl. Phys. **B740** (2006) 299–327, arXiv:math.QA/0601630.
- [13] A. Kuniba and R. Sakamoto, The Bethe ansatz in a periodic box-ball system and the ultradiscrete Riemann theta function, J. Stat. Mech. (2006) P09005, 1–12, arXiv:math.QA/0606208.
- [14] A. Kuniba and R. Sakamoto, Combinatorial Bethe ansatz and ultradiscrete Riemann theta function with rational characteristics, Lett. Math. Phys. **80** (2007) 199–209, arXiv:nlin/0611046.
- [15] A. Kuniba, R. Sakamoto and Y. Yamada, Tau functions in combinatorial Bethe ansatz, Nucl. Phys. **B786** (2007) 207–266, arXiv:math.QA/0610505.
- [16] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd edition, Oxford Univ. Press, New York (1995).
- [17] T. Miwa, M. Jimbo and E. Date, *Solitons: differential equations, symmetries and infinite dimensional algebras*, Cambridge Univ. Press (2000).
- [18] M. Okado, $X = M$ conjecture, MSJ Memoirs **17** (2007) 43–73.

- [19] R. Sakamoto, Crystal interpretation of Kerov-Kirillov-Reshetikhin bijection II. Proof for \mathfrak{sl}_n case, J. Algebraic Combin. **27** (2008) 55–98, arXiv:math.QA/0601697.
- [20] R. Sakamoto, A crystal theoretic method for finding rigged configurations from paths, J. Phys. A: Math. Theor. **41** (2008) 355208 (21pp), arXiv:0708.3544.
- [21] M. Sato and Y. Sato, Soliton equations as dynamical systems on infinite dimensional Grassmann manifold, *Nonlinear PDE in Applied Science, U.S.-Japan Seminar, Tokyo, 1982*, Lecture Notes in Num. Appl. Anal. **5** (1982) 259–271.
- [22] C. Schensted, Longest increasing and decreasing subsequences, Canad. J. Math. **13** (1961) 179–191.
- [23] A. Schilling, Crystal structure on rigged configurations, Int. Math. Res. Notices (2006) Article ID 97376, 1–27, arXiv:math.QA/0508107.
- [24] A. Schilling, $X = M$ Theorem: Fermionic formulas and rigged configurations under review, MSJ Memoirs **17** (2007) 75–104, arXiv:math.QA/0512161.
- [25] A. Schilling and S. O. Warnaar, Inhomogeneous lattice paths, generalized Kostka polynomials and A_{n-1} supernomials, Commun. Math. Phys. **202** (1999) 359–401, arXiv:math/9802111.
- [26] M. Shimozono, Affine type A crystal structure on tensor products of rectangles, Demazure characters, and nilpotent varieties, J. Algebraic Combin. **15** (2002) 151–187, arXiv:math/9804039.
- [27] T. Takagi, Separation of colour degree of freedom from dynamics in a soliton cellular automaton, J. Phys. A: Math. Gen. **38** (2005) 1961–1976, arXiv:math-ph/0411013.
- [28] D. Takahashi, On some soliton systems defined by using boxes and balls, *Proceedings of the International Symposium on Nonlinear Theory and Its Applications (NOLTA '93)*, (1993) 555–558.
- [29] D. Takahashi and J. Satsuma, A soliton cellular automaton, J. Phys. Soc. Japan, **59** (1990) 3514–3519.
- [30] F. Yura and T. Tokihiro, On a periodic soliton cellular automaton, J. Phys. A: Math. Gen. **35** (2002) 3787–3801, arXiv:nlin/0112041.